

A CLASS OF RANDOM FIELDS ON COMPLETE GRAPHS WITH TRACTABLE PARTITION FUNCTION

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Introduction. Ising like models on complete, but *small* graphs are often considered for experiments in papers on new or improved methods for estimating marginal probabilities of random fields [6, 7, 5, 2, 3]. This note aims to draw attention to the fact that the partition function and marginal probabilities can be computed in polynomial time for a certain class of random fields on complete graphs. This class includes Ising models with homogeneous pairwise potentials but arbitrary (inhomogeneous) unary potentials. Similarly, the partition function and marginal probabilities can be computed for random fields on complete bipartite graphs provided they have homogeneous pairwise potentials.

The model class; computing the partition function. Let us consider the following class of binary valued random fields on complete graphs

$$(1) \quad p(x) = \frac{1}{Z} \prod_{ij \in E} g(x_i, x_j) \prod_{i \in V} q_i(x_i),$$

where (V, E) denote the sets of vertices and edges of a complete graph and $x: V \rightarrow \{0, 1\}$ is a binary valued labelling of the vertices. Notice, that we assume that $g: \{0, 1\}^2 \rightarrow \mathbb{R}$ is shared by all pairwise factors. Given the model parameters g and q_i , the task is to compute the partition sum

$$(2) \quad Z = \sum_{x \in \mathcal{X}} \prod_{ij \in E} g(x_i, x_j) \prod_{i \in V} q_i(x_i)$$

as well as unary and pairwise marginal probabilities $p(x_i)$ and $p(x_i, x_j)$. We assume without loss of generality that g has the form

$$(3) \quad g(k, k') = \begin{cases} \alpha & \text{if } k \neq k', \\ 1 & \text{otherwise,} \end{cases}$$

and the unary factors have the form

$$(4) \quad q_i(k) = \begin{cases} \beta_i & \text{if } k = 0, \\ 1 & \text{otherwise.} \end{cases}$$

This can be achieved by applying an appropriate re-parametrisation without changing the probability.

In order to calculate the partition sum Z for a graph with n vertices, we partition the set $\mathcal{X} = \{0, 1\}^n$, of all labellings into the sets $\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_n$, where \mathcal{X}_m denotes the set of all

labellings with m vertices labelled by “1”

$$(5) \quad \mathcal{X}_m = \left\{ x \in \mathcal{X} \mid \sum_{i \in V} x_i = m \right\}.$$

Accordingly, we denote the partial sums by

$$(6) \quad Z_m = \sum_{x \in \mathcal{X}_m} \prod_{ij \in E} g(x_i, x_j) \prod_{i \in V} q_i(x_i).$$

Due to the homogeneity assumption (3), the pairwise factors in (6) contribute to each summand of Z_m by the same factor $\alpha^{m(n-m)}$. Hence, we may write

$$(7) \quad Z = \sum_{m=0}^n Z_m = \sum_{m=0}^n \alpha^{m(n-m)} H_V(m),$$

where

$$(8) \quad H_V(m) = \sum_{x \in \mathcal{X}_m} \prod_{i \in V} q_i(x_i) = \sum_{x \in \mathcal{X}_m} \prod_{i \in V} \beta_i^{1-x_i}$$

denotes the sum of all unary contributions to the partial sum Z_m . These quantities can be computed recursively over the size of the graph. Let us denote by $H_U(m)$, $m = 0, 1, \dots, |U|$, the corresponding quantities for the complete sub-graph induced by the vertex set $U \subset V$. If $i \in U$ is a vertex of this graph, then obviously,

$$(9) \quad H_U(m) = \beta_i H_{U \setminus i}(m) + H_{U \setminus i}(m-1).$$

This results in an algorithm for computing Z with $\mathcal{O}(n^2)$ time complexity.

It is similarly easy to compute marginal probabilities because the mapping defined by (9) is invertible. In order to compute e.g. the unary marginal probabilities for the vertex $i \in V$

$$(10) \quad p(x_i = 1) = \sum_{m=1}^n H_{V \setminus i}(m-1) \alpha^{m(n-m)}$$

$$(11) \quad p(x_i = 0) = \sum_{m=0}^{n-1} H_{V \setminus i}(m) \alpha^{m(n-m)},$$

we need the quantities $H_{V \setminus i}(m)$, which can be computed from those for the whole graph with $\mathcal{O}(n)$ time complexity by

$$(12) \quad H_{V \setminus i}(m) = \frac{1}{\beta_i} \left[H_V(m) - H_{V \setminus i}(m-1) \right].$$

This results in a $\mathcal{O}(n^2)$ algorithm for calculating the unary marginals for *all* vertices of the graph.

The proposed approach can be generalised to K -valued random fields on complete graphs with homogeneous pairwise interactions. For this we partition the set of all labellings \mathcal{X} into sets $\mathcal{X}_{\mathbf{m}}$, where $\mathbf{m} = (m_1, \dots, m_K)$ is a vector, the components of which denote the number of variables taking the respective label value, i.e.,

$$(13) \quad \mathcal{X}_{\mathbf{m}} = \left\{ x \in \mathcal{X} \mid \sum_{i \in V} \delta(x_i, 1) = m_1, \dots, \sum_{i \in V} \delta(x_i, K) = m_K \right\}$$

The corresponding contributions $H_V(\mathbf{m})$ of all unary terms to the partial sums can be again computed recursively by

$$(14) \quad H_U(\mathbf{m}) = \sum_{k=1}^K q_i(k) H_{U \setminus i}(\mathbf{m} - \mathbf{e}_k),$$

where \mathbf{e}_k denotes the standard basis vector for the component k . This leads to the following theorem.

Theorem 1. *Given a K -valued random field on a complete graph with homogeneous pairwise factors and arbitrary (inhomogeneous) unary factors. Its partition sum can be computed with $\mathcal{O}(n^K)$ time complexity.*

Remark 1. Note, that the theorem is not in contradiction with the dichotomy found by Bulatov and Grohe [1] because here we restrict the graph structure.

Models on complete bipartite graphs. The applicability of the proposed approach can be extended even further, e.g. for the calculation of the partition function and marginal probabilities for random fields defined on complete bipartite graphs. Here, again, it is required that the pairwise interactions are homogeneous. We will consider binary valued random fields for the sake of simplicity.

Let $G(V, E)$ be a complete bipartite graph and V_1, V_2 denote its parts ($n_1 = |V_1|, n_2 = |V_2|$). Consider the class of binary valued random fields $x: V \rightarrow \{0, 1\}$

$$(15) \quad p(x) = \frac{1}{Z} \prod_{\substack{i \in V_1 \\ j \in V_2}} g(x_i, x_j) \prod_{i \in V} q_i(x_i).$$

As discussed in the previous section, we may assume that the factors g, q_i have the form (3) and (4) respectively.

In order to compute Z , we partition the set of all labellings \mathcal{X} into the sets

$$(16) \quad \mathcal{X}_{\mathbf{m}} = \left\{ x \in \mathcal{X} \mid \sum_{i \in V_1} x_i = m_1, \sum_{j \in V_2} x_j = m_2 \right\},$$

where $\mathbf{m} = (m_1, m_2)$ counts the number of vertices labelled by “1” in the first and second part, respectively. It is clear that the pairwise factors contribute to all summands of $Z_{\mathbf{m}}$ by the same factor α^κ , where $\kappa = m_1(n_2 - m_2) + (n_1 - m_1)m_2$. Hence, as before, the problem reduces to the computation of the contributions $H_{V_1, V_2}(\mathbf{m})$ of the unary factors. They can be computed recursively over the graph size. Similar to (9) we have here

$$(17) \quad H_{U_1, U_2}(\mathbf{m}) = \beta_i \cdot H_{U_1 \setminus i, U_2}(\mathbf{m}) + H_{U_1 \setminus i, U_2}(\mathbf{m} - \mathbf{e}_1),$$

$$(18) \quad H_{U_1, U_2}(\mathbf{m}) = \beta_j \cdot H_{U_1, U_2 \setminus j}(\mathbf{m}) + H_{U_1, U_2 \setminus j}(\mathbf{m} - \mathbf{e}_2).$$

This results in a $\mathcal{O}(n_1^2 n_2^2)$ algorithm for computing the partition sum Z . Generalising this to K -valued labellings as in the previous section we get the following result.

Theorem 2. *Given a K -valued random field on a complete bipartite graph with homogeneous pairwise factors and arbitrary (inhomogeneous) unary factors (15). Its partition sum can be computed with $\mathcal{O}(n_1^K n_2^K)$ time complexity.*

Conclusion. We have shown that the partition sum and marginal probabilities can be efficiently computed for random fields on complete graphs if they have homogeneous pairwise factors. Similarly, the partition sum and marginal probabilities can be efficiently computed for random fields on complete bipartite graphs provided they have homogeneous pairwise factors. We do not expect these two model classes to be directly relevant for computer vision applications. We expect, however, that they can be very useful to evaluate approximation algorithms for computing marginal probabilities. To the best of our knowledge they are the only so far known classes of *large scale* random fields defined on graphs with large tree-width and with arbitrary unary factors¹ for which the marginal probabilities can be computed in polynomial time, thus providing exact error estimates for approximation algorithms.

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¹The method presented in [4] requires outer-planar graphs, i.e. tree-width two, if applied for the case of arbitrary unary factors.